THE RING STRUCTURE ON THE GROTHENDIECK GROUP OF COMMUTING ENDOMORPHISMS OVER A FIELD

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Abstract. For a perfect field k we compute the ring structure on the Grothendieck group of the category of finitely many commuting endomorphisms of finite-dimensional vector spaces over a field.

1. Introduction

Let k be a commutative ring and $\mathcal{P}(k)$ the category of finitely-generated projective k-modules. Let $\operatorname{End}\mathcal{P}(k)$ be the category whose objects are endomorphisms $P \to P$ where $P \in \mathcal{P}(k)$, and whose morphisms $\alpha : (P \to P) \to (Q \to Q)$ are morphisms $\alpha : P \to Q$ in $\mathcal{P}(k)$ such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & Q \\ \downarrow & & \downarrow \\ P & \xrightarrow{\alpha} & Q \end{array}$$

commutes. The category $\operatorname{End}\mathcal{P}(k)$ is naturally equivalent to the category of $k[t]$-modules that are finitely generated and projective as k-modules via the inclusion map $k \to k[t]$. The category $\operatorname{End}\mathcal{P}(k)$ is an exact category and one would like to calculate its Quillen K-theory groups. This type of K-theory calculation falls under the more general problem of calculating for a ring homomorphism $R \to S$, the K-theory of the category of S-modules that are finitely generated and projective as R-modules.

The calculation of $K_i(\operatorname{End}\mathcal{P}(k))$ was given by Kelley and Spanier [KS68] when k is a field and $i = 0$. Almkvist [Alm78] did the calculation when k is an arbitrary commutative ring and $i = 0$, and when k is a field and i is arbitrary. To describe these computations, define the abelian group whose underlying set is

$$\tilde{k}_0 = \left\{ \frac{1 + a_1 t + \cdots + a_n t^n}{1 + b_1 t + \cdots + b_m t^m} : a_i, b_j \in k \right\}$$

and whose binary operation is given by the usual multiplication of rational functions. If $f : M \to M$ is an endormorphism of a finitely-generated projective k-module, then the characteristic polynomial $\lambda_t(f)$ may be defined by extending f to an endomorphism of a free module, and then defining $\lambda_t(f) = \det(1+tf)$; see [Alm78] for an alternative definition.

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The split inclusion k-$\mathbf{mod_{fg}} \to \operatorname{End}\mathcal{P}(k)$ induces a corresponding splitting

$$K_0(\operatorname{End}\mathcal{P}(k)) \cong K_0(k) \times \tilde{K}_0(\operatorname{End}\mathcal{P}(k)).$$

Then we have:

1.1. **Theorem** (Kelley-Spanier, Almkvist)**.** *There is an isomorphism*

$$\tilde{K}_0(\operatorname{End}\mathcal{P}(k)) \longrightarrow \tilde{k}_0$$
$$[M \xrightarrow{f} M] \longmapsto \lambda_t(f).$$

The group $K_0(\operatorname{End}\mathcal{P}(k))$ has a natural ring structure, as described by Kelley and Spanier. Given two $k[t]$-modules M and N, the tensor product $M \otimes_k N$ has a natural k-module structure. One can define an action of t also, by declaring that

$$t(m \otimes n) = tm \otimes tn$$

for pure tensors $m \otimes n$. This gives $M \otimes_k N$ an $k[t]$-module structure. This construction gives a multiplication $*$ on $K_0(\operatorname{End}\mathcal{P}(k))$, which corresponds on generators to

$$(M, f) * (N, g) = (M \otimes_k N, f \otimes g).$$

Summary of This Paper. The ring structure on $\tilde{K}_0(\operatorname{End}\mathcal{P}(k))$ was determined by Almkvist for $k = \mathbb{R}$ and $k = \mathbb{C}$. In this current work, we describe the ring structure for *any* perfect field k: let $\operatorname{End}_n\mathcal{P}(k)$ denote the analogous category of n-commuting endomorphisms of finite-dimensional k-vector spaces. Then:

1.2. **Theorem.** *If k is a perfect field then there is an isomorphism of rings* $\tilde{K}_0(\operatorname{End}_n\mathcal{P}(k)) \cong \mathbb{Z}[(\bar{k}^\times)^n]^{\operatorname{Gal}(\bar{k}/k)}$.

This generalises Almkvist's result to arbitrary perfect fields and to n-commuting endomorphisms. To make our exposition more self-contained, in §2, we review the calculation of $\tilde{K}_i(\operatorname{End}_n\mathcal{P}(k))$ as an abelian group. Although this result is well-known, to our knowledge this proof is not available in such a self-contained way and thus the presentation should be quite helpful for beginners. The proof of Theorem 1.2 is then presented in §3. We also pose two open questions that naturally stem from this work.

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2. The Category $\operatorname{End}_n\mathcal{P}(k)$

Let $\operatorname{End}_n\mathcal{P}(k)$ denote the exact category of $k[T] := k[t_1, \ldots, t_n]$-modules that are finitely generated as k-modules. *In this section*, we calculate the abelian group $K_i(\operatorname{End}_n\mathcal{P}(k))$. A finite-dimensional k-vector space V with any n commuting endomorphisms $f_1, f_2, \ldots, f_n$ of V may also be considered as a $k[t_1, \ldots, t_n]/I$ module where I is the kernel of the map $k[t_1, \ldots, t_n] \to k[f_1, \ldots, f_n]$ given by $t_i \mapsto f_i$; hence we are led naturally to:

2.1. **Proposition.** *The category* $\operatorname{End}_n\mathcal{P}(k)$ *is naturally equivalent to the filtered direct limit*

$$\varinjlim_I \, (k[T]/I\text{-}\mathbf{mod_{fg}}) \tag{1}$$

of exact categories, where I runs over the set of ideals of $k[T]$ such that the quotient $k[T]/I$ is finite-dimensional over k, and if $I \subseteq J$ are two such ideals, then the functor $k[T]/J\text{-}\mathbf{mod_{fg}} \to k[T]/I\text{-}\mathbf{mod_{fg}}$ is the forgetful functor induced by the quotient map $k[T]/I \to k[T]/J$.

Proof. The limit is indeed filtered, since $k[T]/(I \cap J)$ is finite-dimensional whenever $k[T]/I$ and $k[T]/J$ are finite-dimensional. Define a functor

$$F : \varinjlim_I (k[T]/I\text{-}\mathbf{mod_{fg}}) \to \operatorname{End}_n \mathcal{P}(A)$$

as follows. Any element in $\varinjlim_I (k[T]/I\text{-}\mathbf{mod_{fg}})$ is represented by $V \in k[T]/I\text{-}\mathbf{mod_{fg}}$ for some I. We let $F(V)$ to be the $k[T]$ module given by the forgetful functor induced by the map $k[T] \to k[T]/I$. This functor is well-defined because $k[T]/I$ is finite-dimensional, and it is easy to see that F gives the required natural equivalence. □

Using this observation and the result that taking K-groups of exact categories commutes with filtered direct limits [Qui72, §2], we obtain the following corollary.

2.2. **Corollary.** *The K-theory of the category $\operatorname{End}_n \mathcal{P}(A)$ may be calculated as the direct limit*

$$K_i(\operatorname{End}_n \mathcal{P}(A)) \cong \varinjlim_I K_i\left(k[T]/I\text{-}\mathbf{mod_{fg}}\right) \tag{2}$$

For any ring R, let $\operatorname{Jac}(R)$ denote the Jacobson radical of R. Any surjective ring homomorphism $R \to S$ induces a surjective ring homomorphism $R/\operatorname{Jac}(R) \to S/\operatorname{Jac}(S)$, and a commutative diagram of rings

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ R/\operatorname{Jac}(R) & \longrightarrow & S/\operatorname{Jac}(S) \end{array}$$

In turn, whenever S is finitely generated as an R-module, we have a commutative diagram of forgetful functors

$$\begin{array}{ccc} R\text{-}\mathbf{mod_{fg}} & \longleftarrow & S\text{-}\mathbf{mod_{fg}} \\ \uparrow & & \uparrow \\ R/\operatorname{Jac}(R)\text{-}\mathbf{mod_{fg}} & \longleftarrow & S/\operatorname{Jac}(S)\text{-}\mathbf{mod_{fg}} \end{array} \tag{3}$$

In particular, this applies to the ring homomorphisms $k[T]/I \to k[T]/J$ for $I \subseteq J$ appearing in the direct limit of (1).

2.3. **Proposition.** *The natural transformation of the direct limits of categories induced by the forgetful functors as in (3) for $R = k[T]/I$ induces an isomorphism algebraic K-groups*

$$\varinjlim_I K_i\left(\frac{k[T]/I}{\operatorname{Jac}(k[T]/I)}\text{-}\mathbf{mod_{fg}}\right) \xrightarrow{\sim} \varinjlim_I K_i\left(k[T]/I\text{-}\mathbf{mod_{fg}}\right).$$

Proof. For any Artinian ring R, the Jacobson radical $\operatorname{Jac}(R)$ is nilpotent (e.g. [Lam91, Theorem 4.12]), and so devissage [Qui72, §5, Theorem 4] shows that the inclusion $R/\operatorname{Jac}(R)\text{-}\mathbf{mod_{fg}} \to$

R-$\mathbf{mod_{fg}}$ induces an isomorphism

$$K_i(R/\operatorname{Jac}(R)\text{-}\mathbf{mod_{fg}}) \to K_i(R\text{-}\mathbf{mod_{fg}})$$

(see [Wei13, Page 439] for more details). In particular, this applies to $R = k[T]/I$ where $k[T]/I$ is finite-dimensional over k. ☐

2.4. **Theorem.** *The algebraic K-groups of the exact category* $\operatorname{End}_n \mathcal{P}(k)$ *are given by*

$$K_i(\operatorname{End}_n \mathcal{P}(k)) \cong \bigoplus_M K_i(k[t_1, \dots, t_n]/M) \tag{4}$$

where M ranges over all the maximal ideals of the polynomial ring $k[t_1, \dots, t_n]$.

Proof. We must calculate the limit

$$\varinjlim_I K_i\left(\frac{k[T]/I}{\operatorname{Jac}(k[T]/I)}\text{-}\mathbf{mod_{fg}}\right) \tag{5}$$

given in Proposition 2.3. So, fix an ideal I such that $k[T]/I$ is finite-dimensional. Then there are finitely many maximal ideals $M_1, \dots, M_k$ of $k[T]$ that contain I and

$$k[T]/I \cong \bigoplus_{j=1}^{k} (k[T]/I)_{M_j}$$

via the obvious map (e.g. [GW10, Theorem 5.20]). Here, by $(k[T]/I)_{M_i}$, we abuse notation and mean the localization of $k[T]/I$ away from the maximal ideal $M_i(k[T]/I)$. If J is an ideal containing I, then there is a subset $\{N_1, \dots, N_\ell\}$ of the maximal ideals $\{M_1, \dots, M_k\}$ that contain J and the quotient homomorphism $k[T]/I \to k[T]/J$ induces the map

$$\bigoplus_{j=1}^{k} (k[T]/I)_{M_j} \longrightarrow \bigoplus_{j=1}^{\ell} (k[T]/J)_{N_j}$$

which must be the projection map. The induced map on K-groups that fits into the direct limit in (5), by is then the map

$$\bigoplus_{j=1}^{\ell} K_i(k[T]/N_j) \longrightarrow \bigoplus_{j=1}^{k} K_i(k[T]/M_j).$$

given by the sum of the inclusion maps. Taking the direct limit gives the stated result, once we note that $k[T]/M$ is finite-dimensional over k for any maximal ideal M. ☐

In particular, we obtain the following amusing and well-known result.

2.5. **Corollary.** *For a field k, there is an isomorphism of abelian groups*

$$\tilde{k}_0 := \left\{ \frac{1 + a_1 t + \cdots + a_n t^n}{1 + b_1 t + \cdots + b_m t^m} : a_i, b_j \in k \right\} \cong \bigoplus_{|\operatorname{Spec}(k[T])|-1} \mathbb{Z}.$$

3. The Ring Structure For One Endomorphism

In this section, we determine the $\tilde{K}_0(\operatorname{End}_n \mathcal{P}(k))$ for perfect fields, generalising the known cases of the real numbers and algebraically closed fields. To start out investigation however, we give an alternative proof of the following theorem for algebraically closed fields:

3.1. **Proposition** ([Alm78, Proposition 3.4, Part 1])**.** *If k is an algebraically closed field then*

$$\tilde{K}_0(\operatorname{End} \mathcal{P}(k)) \cong \mathbb{Z}[k^\times]$$

where $\mathbb{Z}[k^\times]$ denotes the integral group ring for the multiplicative group $k^\times$ of k.

Proof. The group $\tilde{K}_0(\operatorname{End} \mathcal{P}(k))$ is the free abelian group $\bigoplus_{a\in k^\times} K_0(k[t]/(t-a))$. Suppose $a, b \in k^\times$. The generators for the direct summands corresponding to $K_0(k[t]/(t-a))$ and $K_0(k[t]/(t-b))$ are $[k]$ where t acts on k by a and by b respectively. The multiplication $[k] * [k]$ is then represented by $k \otimes_k k \cong k$ where t acts by ab, and so is the element corresponding to the generator $[k]$ for the summand $K_0(k[t]/(t-ab))$. Hence the map

$$\bigoplus_{a\in k^\times} \mathbb{Z} \longrightarrow \mathbb{Z}[k^\times]$$
$$(z_a)_{a\in k^\times} \longmapsto \sum_{a\in k^\times} z_a[a]$$

extends to an isomorphism $\tilde{K}_0(\operatorname{End} \mathcal{P}(k)) \to \mathbb{Z}[k^\times]$ as required. □

The above proof also shows that for any field k, the group ring $\mathbb{Z}[k^\times]$ is a subring of $\tilde{K}_0(\operatorname{End} \mathcal{P}(k))$, and the elements of this subring make up exactly the subgroup of $\tilde{K}_0(\operatorname{End} \mathcal{P}(k))$ consisting of those summands corresponding to the maximal ideals generated by polynomials of degree one. In particular, the ring structure on $\tilde{K}_0(\operatorname{End} \mathcal{P}(k))$ makes $\tilde{K}_0(\operatorname{End} \mathcal{P}(k))$ into a $\mathbb{Z}[k^\times]$-module.

3.2. **Proposition.** *The action of the ring $\mathbb{Z}[k^\times]$ on the abelian group $\tilde{K}_0(\operatorname{End} \mathcal{P}(k))$ making $\tilde{K}_0(\operatorname{End} \mathcal{P}(k))$ into a $\mathbb{Z}[k^\times]$-module can be described on generators by*

$$\left(a, \left[\frac{k[t]}{t^n - a_{n-1}t^{n-1} - \cdots - a_1 t - a_0}\right]\right) \mapsto \left[\frac{k[t]}{t^n - aa_{n-1}t^{n-1} - \cdots - a^{n-1}a_1 t - a^n a_0}\right]$$

for all $a \in k^\times$.

Proof. Indeed, the chosen representative on the right is isomorphic as a $k[t]$-module to

$$\frac{k[t]}{t-a} \otimes_k \frac{k[t]}{t^n - a_{n-1}t^{n-1} - \cdots - a_1 t - a_0}.$$

□

It is easy to see that $t^n - aa_{n-1}t^{n-1} - \cdots - a^{n-1}a_1 t - a^n a_0$ is irreducible whenever $t^n - a_{n-1}t^{n-1} - \cdots - a_1 t - a_0$ is irreducible, so that elements of the group $k^\times$ act as shift operators on $\tilde{K}_0(\operatorname{End} \mathcal{P}(k))$.

At this point, it seems more natural to work with the ring $\tilde{k}_0$ directly, though to be consistent with our description in terms of maximal ideals as in Theorem 2.4, we will use the $f \mapsto \det(\mathbf{1}t - f)$ version of the characteristic polynomial. So, we will abuse notation and write $\tilde{k}_0$ to mean

$$\tilde{k}_0 = \left\{ \frac{t^n - a_{n-1}t^{n-1} - \cdots - a_0}{t^m - b_{m-1}t^{m-1} - \cdots - b_0} : (a_i) \in (k^\times)^n, (b_i) \in (k^\times)^n \right\}.$$

Then, Proposition 3.2 tells us that the multiplication in $\tilde{k}_0$ of an element of the form $t - a$ with an arbitrary element can be described on generators by

$$(t - a) * (t^n - a_{n-1}t^{n-1} - \cdots - a_0) = t^n - aa_{n-1}t^{n-1} - \cdots - a^n a_0. \tag{6}$$

Naturally, with formula (6), one can also multiply higher degree generators together if one of them factors completely over k, since *usual polynomial multiplication* is addition in the ring $\tilde{k}_0$. One could define the product $f * g$ of two polynomials in $\tilde{k}_0$ by factoring one over a finite extension of k, and perform the multiplication there. Indeed, for any field k and an automorphism $\sigma : k \to k$ of fields, it is clear that $\sigma(f * g) = \sigma(f) * \sigma(g)$. Hence $f * g$ obtained in this way is still an element of $\tilde{k}_0$. This observation leads us to:

3.3. **Theorem.** *For any field k, there is an isomorphism $\tilde{K}_0(\operatorname{End}\mathcal{P}(k)) \cong \tilde{k}_0$, where $\tilde{k}_0$ is the ring whose underlying abelian group is free on the irreducible monic polynomials $f \in k[t]$ such that $f \neq t$, and whose multiplication is given for generators f and g by:*

(1) *If $f = t - a$ where $a \in k^\times$ and $g = t^n - a_{n-1}t^{n-1} - \cdots - a_0$ then*

$$f * g = t^n - aa_{n-1}t^{n-1} - \cdots - a^n a_0,$$

(2) *In general, if $f(t) = (t - e_1)(t - e_2)\cdots(t - e_k)$ where $(e_i) \in (E^\times)^n$ and E is a splitting field for f, then*

$$f * g = [(t - e_1) * g][(t - e_2) * g] \cdots [(t - e_k) * g].$$

Proof. The multiplication in (1) is Proposition 3.2. The multiplication rule in (2) follows because for each field extension E/k, the base change functor

$$\begin{aligned} \operatorname{End}\mathcal{P}(k) &\longrightarrow \operatorname{End}\mathcal{P}(E) \\ (V, f) &\longmapsto (V \otimes_k E, f \otimes 1) \end{aligned}$$

induces a homomorphism of rings when passing to $\tilde{K}_0$ that commutes with the isomorphism $\tilde{K}_0(\operatorname{End}\mathcal{P}(k)) \cong \tilde{k}_0$, and hence one can multiply after base change to a splitting field, using (1) for that splitting field. □

3.4. *Example.* If $k = \mathbb{F}_2(t)$ then in $\tilde{k}_0$, we have $(x^2 + t) * (x^2 + t) = (x + t)^4$.

3.5. **Corollary.** *If k is perfect then there is an isomorphism of rings $\tilde{k}_0 \cong \mathbb{Z}[\bar{k}^\times]^{\mathrm{Gal}(\bar{k}/k)}$.*

Proof. An isomorphism is defined on generators by sending an irreducible monic polynomial f with factorisation $f(t) = (t - a_1)(t - a_2)\dots(t - a_n)$ over $\bar{k}$ to the element $\sum_{i=1}^n a_i \in \mathbb{Z}[\bar{k}^\times]^{\mathrm{Gal}(\bar{k}/k)}$. This is a ring homomorphism by Theorem 3.3, and it is bijective because over a perfect field, irreducible monic polynomials $f \in k[t]$ with $f(0) \neq 0$ correspond to the $\mathrm{Gal}(\bar{k}/k)$-orbits in $\bar{k}^\times$. □

Corollary 3.5 suggests the multivariable analogue. Recall that we use the notation $k[T] := k[t_1, \ldots, t_n]$, and we let $\tilde{K}_0(\operatorname{End}_n \mathcal{P}(k))$ denote the nontrivial part of the group $K_0(\operatorname{End}_n \mathcal{P}(k))$. In other words, $\tilde{K}_0(\operatorname{End}_n \mathcal{P}(k))$ is the Grothendieck group of the category of n-commuting endomorphisms, none of which are zero. The following proposition is well-known.

3.6. **Proposition.** *Let k be a perfect field. If $M \subset k[T]$ is a maximal ideal, then $M\bar{k}[T] = M_1 \cap \cdots \cap M_d$ where $\{M_1, \ldots, M_d\}$ is the set of maximal ideals lying over M in $\bar{k}[T]$. The map*

$$M \mapsto \{M_1, \ldots, M_d\}$$

is a bijection between the maximal ideals of $k[T]$ and the Galois orbits in the set of maximal ideals of $\bar{k}[T]$.

Proof. This follows from [Mat86, Theorem 9.3,(iii)] and the fact that if $\{M_1, \ldots, M_d\}$ is a Galois orbit, then $(M_1 \cap \cdots \cap M_k) \cap k[T]$ is a maximal ideal of $k[T]$. □

3.7. **Theorem.** *If k is a perfect field then there is an isomorphism of rings $\tilde{K}_0(\operatorname{End}_n \mathcal{P}(k)) \cong \mathbb{Z}[(\bar{k}^\times)^n]^{\operatorname{Gal}(\bar{k}/k)}$.*

Proof. A generator of $\tilde{K}_0(\operatorname{End}_n \mathcal{P}(k))$ will be of the form $[k[T]/M]$ for some maximal ideal M. Proposition 3.6 says that $M\bar{k}[T] = \cap_{i=1}^d M_i$ where $\{M_1, \ldots, M_k\}$ are the maximal ideals lying over M. By Hilbert's Nullstellensatz each of these ideals are of the form

$$M_i = (t_1 - a_{i1}, \ldots, t_n - a_{in})$$

where $a_{ij} \in \bar{k}$. The map

$$\tilde{K}_0(\operatorname{End}_n \mathcal{P}(k)) \longrightarrow \mathbb{Z}[(\bar{k}^\times)^n]^{\operatorname{Gal}(\bar{k}/k)}$$

is then defined on generators by

$$[k[T]/M] \longmapsto \sum_{i=1}^{d} (a_{i1}, \ldots, a_{in}).$$

This is a ring homomorphism as in the Grothendieck group, direct sum and tensor product correspond to addition and multiplication, and Proposition 3.6 shows that it is a bijection. □

3.8. *Remark.* For imperfect fields, the maps to the Galois-invariant elements of group rings are still well-defined, but they are no longer surjective.

We would like to end this paper with two natural important open questions that stem from this work, the first being to compute a Grothendieck group and the second being an inverse or categorification problem:

3.9. *Open Question.* If k is a commutative ring, what is the ring structure on $\tilde{K}_0(\operatorname{End}_n(k))$?

3.10. *Open Question.* If k is a commutative ring with no nontrivial idempotents, then there exists a Galois theory for k analogous to fields and a separable closure $\bar{k}$. In this case, is there an appropriate category whose Grothendieck group can be given a ring structure such that the resulting ring is isomorphic to $\mathbb{Z}[(\bar{k})^\times]^{\operatorname{Gal}(\bar{k}/k)}$?

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