A COTRIPLE CONSTRUCTION OF A SIMPLICIAL ALGEBRA USED IN THE DEFINITION OF HIGHER CHOW GROUPS

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ABSTRACT. We present a brief and simple cotriple description of the simplicial algebra used in Bloch's construction of the higher Chow groups.

1. A Simplicial Algebra

Let Δ be the category of finite ordered sets and nondecreasing functions. For each natural number n, we let $[n]$ denote the object $\{0 < \cdots < n\}$ of Δ. Let R be a commutative ring and let

$$D[n] = R[t_0, \dots, t_n] / \left(\sum t_i - 1 \right).$$

For each $\rho : [m] \to [n]$, we define $D[n] \to D[m]$ as the R-algebra homomorphism

$$D(\rho) : \frac{R[t_0, \dots, t_n]}{\sum t_i - 1} \longrightarrow \frac{R[t_1, \dots, t_m]}{\sum t_i - 1}$$
$$t_i \longmapsto \sum_{\rho(j)=i} t_j.$$

The cosimplicial scheme $[n] \mapsto \mathrm{Spec}(D[n])$ arises in Bloch's definition of higher Chow groups [Blo86]. Given this definition, one needs to go through a straightforward but somewhat tedious checking that such a simplicial R-algebra is actually simplicial, which involves verifying a few identities.

In this paper an alternative way to construct the simplicial R-algebra as above in terms of a cotriple that additionally obviates the need to check simplicial identities. In §2 we review the basic notions involving simplicial objects and cotriples, and in §3 we present our construction.

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2. Simplicial Objects and Cotriples

The material in this section is standard and we merely review it to fix notation. The reader should consult [Wei94] or [BW00] for more details. A **simplicial object** A with values in a category $\mathcal{C}$ is a contravariant functor $A : \Delta \to \mathcal{C}$. We define the map $\varepsilon_i : [n-1] \to [n]$ to be the unique map whose image does not contain $i \in [n]$ and $\eta_i : [n+1] \to [n]$ to be the unique map such that two elements of $[n+1]$ map to i. The **face maps** of a simplicial object A are defined to be $\partial_i := A(\varepsilon_i)$ and the **degeneracy maps** are defined to be $\sigma_i := A(\eta_i)$. We recall that a simplicial object is determined by the ∂_i and σ_i maps.

Given a simplicial object A, the **dual** $A^\vee$ to A is the simplicial object that is the same as A except that $\partial_i^\vee = \partial_{n-i}$ and $\sigma_i^\vee = \sigma_{n-i}$.

Definition 2.1. *A **cotriple** $(\perp, \varepsilon, \delta)$ on a category $\mathcal{C}$ is a functor $\perp : \mathcal{C} \to \mathcal{C}$ together with two natural transformations $\varepsilon : \perp \to \mathbf{1}_{\mathcal{C}}$ and $\delta : \perp \to \perp\perp$ such that for every object X in $\mathcal{C}$, the diagrams*

$$\begin{array}{ccc} \perp X & \xrightarrow{\delta_X} & \perp(\perp X) \\ \downarrow{\scriptstyle \delta_X} & & \downarrow{\scriptstyle \delta_{\perp X}} \\ \perp(\perp X) & \xrightarrow{\perp\delta_X} & \perp\perp\perp X \end{array}$$

$$\begin{array}{ccccc} & \overset{=}{\swarrow} & \perp X & \overset{=}{\searrow} & \\ & & \downarrow{\scriptstyle \delta_X} & & \\ \perp X & \xleftarrow[\perp\varepsilon_X]{} & \perp(\perp X) & \xrightarrow[\varepsilon_{\perp X}]{} & \perp X \end{array}$$

commute.

One may construct simplicial objects out of a cotriple using the following:

Proposition 2.2. *If $(\perp, \varepsilon, \delta)$ is a cotriple in $\mathcal{C}$, then one can construct a simplicial object $[n] \mapsto \perp_n X$ for each object $X \in \mathcal{C}$ by setting $\perp_n X := \perp^{n+1} X$ and*

$$\partial_i := \perp^i \varepsilon \perp^{n-i} : \perp_n X \to \perp_{n-1} X,$$
$$\sigma_i := \perp^i \delta \perp^{n-i} : \perp_n X \to \perp_{n+1} X.$$

3. The Construction

Let R be a commutative ring. We write R-$\mathbf{Alg}_*$ for the category of *pointed* R-algebras. The objects of R-$\mathbf{Alg}_*$ are pairs (A, a) where A is an R-algebra and $a \in A$. A morphism $f : (A, a) \to (B, b)$ in R-$\mathbf{Alg}_*$ is an R-algebra homomorphism $f : A \to B$ such that $f(a) = b$.

Definition 3.1. *We define the functor* $\perp : R\text{-}\mathbf{Alg}_* \to R\text{-}\mathbf{Alg}_*$ *on objects by* $\perp(A, a) = (A[t], t + a)$*, and on morphisms* $f : (A, a) \to (B, b)$ *by*

$$\perp(f)(a_0 + \cdots + a_n t^n) = f(a_0) + \cdots + f(a_n)t^n.$$

We introduce two natural transformations $\varepsilon : \perp \to \mathbf{1}$ and $\delta : \perp \to \perp\perp$ given for each pointed R-algebra (A, a) by

$$\begin{aligned} \varepsilon_A : (A[t], t + a) &\longrightarrow (A, a) \\ t &\longmapsto 0 \end{aligned}$$

and

$$\begin{aligned} \delta_A : (A[t], t + a) &\longrightarrow (A[t, s], s + t + a) \\ t &\longmapsto s + t. \end{aligned}$$

We note we have abused notation by writing ε_A and δ_A when the notation $\varepsilon_{(A,a)}$ and $\delta_{(A,a)}$ would be more accurate and more horrible as well.

Theorem 3.2. *The tuple* $(\perp, \varepsilon, \delta)$ *is a cotriple.*

Proof. We need to verify the commutativity of two diagrams. Let (A, a) be an arbitrary object of R-$\mathbf{Alg}_*$. The first diagram is

$$\begin{array}{ccc} (A[t], t + a) & \xrightarrow{\delta_A} & (A[t, s], s + t + a) \\ \big\downarrow{\scriptstyle \delta_A} & & \big\downarrow{\scriptstyle \delta_{\perp A}} \\ (A[t, s], s + t + a) & \xrightarrow{\perp\delta_A} & (A[u, t, s], s + t + u + a) \end{array}$$

The clockwise direction corresponds to $t \mapsto s + t \mapsto (s + u) + t$, whereas the counterclockwise direction corresponds to $t \mapsto s + t \mapsto s + (t + u)$. The second diagram is similar: it corresponds to $s + t \mapsto 0 + t \mapsto 0 + 0$ in one direction, and $s + t \mapsto s + 0 \mapsto 0 + 0$ in the other. $\square$

Since $(\perp, \varepsilon, \delta)$ is a cotriple, for each (A, a) there is a corresponding simplicial object $\perp_n(A, a)$. To recover Bloch's simplicial algebra, apply the cotriple to the object $(R, -1)$. Then $\perp_n(R, -1) = (R[t_0, \ldots, t_n], \sum t_i - 1)$. The face operator ∂_{n-i} is then

$$\partial_{n-i}(t_j) = \begin{cases} t_j & \text{if } j < i \\ 0 & \text{if } j = i \\ t_{j-1} & \text{if } j > i. \end{cases}$$

and the degeneracy operator σ_{n-i} is the map

$$\sigma_{n-i}(t_j) = \begin{cases} t_j & \text{if } j < i \\ t_j + t_{j+1} & \text{if } j = i \\ t_{j+1} & \text{if } j > i \end{cases}$$

Theorem 3.3. *Let $Q : R\text{-}\mathbf{Alg}_* \to R\text{-alg}$ be the functor to R-algebras defined by $Q(A, a) = A/(a)$. Then Bloch's simplicial algebra D is isomorphic to $Q(\perp_*(R, -1))^\vee$.*

Proof. The ith face map of D comes from applying the functor D to the map ε_i:

$$D(\varepsilon_i)(t_j) = \sum_{\varepsilon_i(j)=k} t_k$$

which one sees gives the same *formula* as the formula for ∂_{n-i}. Similarly, the ith degeneracy map of D comes from applying D to the map η_i and we see that $D(\eta_i)$ is given by the same formula as the σ_{n-i} map as above. □

Although the construction is straightforward, it is arguably a more natural and categorical approach to our simplicial algebra. We remark that there is no obvious adjunction lying around that gives the cotriple we have constructed, even though many standard or obvious cotriples do come from adjunctions.

References

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