THE RING STRUCTURE ON THE GROTHENDIECK GROUP OF ENDOMORPHISMS OVER A FIELD

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ABSTRACT. For a field k we describe the ring structure on the Grothendieck group of endomorphisms of finite-dimensional vector spaces over a field.

1. INTRODUCTION

Let A be a commutative ring and $\mathcal{P}(A)$ the category of finitely-generated projective A-modules. Let $\operatorname{End}\mathcal{P}(A)$ be the category whose objects are endomorphisms $P \to P$ where $P \in \mathcal{P}(A)$, and whose morphisms $\alpha : (P \to P) \to (Q \to Q)$ are morphisms $\alpha : P \to Q$ in $\mathcal{P}(A)$ such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & Q \\ \downarrow & & \downarrow \\ P & \xrightarrow{\alpha} & Q \end{array}$$

commutes. The category $\operatorname{End}\mathcal{P}(A)$ is naturally equivalent to the category of $A[t]$-modules that are finitely generated and projective as A-modules via the inclusion map $A \to A[t]$. The category $\operatorname{End}\mathcal{P}(A)$ is an exact category and one would like to calculate its Quillen K-theory groups. This type of K-theory calculation falls under the more general problem of calculating for a ring homomorphism $R \to S$, the K-theory of the category of S-modules that are finitely generated and projective as R-modules.

The calculation of $K_i(\operatorname{End}\mathcal{P}(A))$ was given by Kelley and Spanier [KS68] when A is a field and $i = 0$. Almkvist [Alm78] did the calculation when A is an arbitrary commutative ring and $i = 0$, and when A is a field and i is arbitrary. To describe these computations, define the abelian group whose underlying set is

$$\tilde{A}_0 = \left\{ \frac{1 + a_1 t + \cdots + a_n t^n}{1 + b_1 t + \cdots + b_m t^m} : a_i, b_j \in A \right\}$$

and whose binary operation is given by the usual multiplication of rational functions. If $f : M \to M$ is an endormorphism of a finitely-generated projective A-module, then the characteristic polynomial $\lambda_t(f)$ may be defined by extending f to an endomorphism of a free module, and then defining $\lambda_t(f) = \det(1 + tf)$; see [Alm78] for an alternative definition.

The split inclusion A-$\mathbf{mod_{fg}} \to \operatorname{End}\mathcal{P}(A)$ induces a corresponding splitting

$$K_0(\operatorname{End}\mathcal{P}(A)) \cong K_0(A) \times \tilde{K_0}(\operatorname{End}\mathcal{P}(A)).$$

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Then we have:

1.1. **Theorem** (Kelley-Spanier, Almkvist)**.** *There is an isomorphism*

$$\tilde{K}_0(\operatorname{End}\mathcal{P}(A)) \longrightarrow \tilde{A}_0$$

$$[M \xrightarrow{f} M] \longmapsto \lambda_t(f).$$

The group $K_0(\operatorname{End}\mathcal{P}(k))$ has a natural ring structure, as described by Kelley and Spanier. Given two $k[t]$-modules M and N, the tensor product $M \otimes_A N$ has a natural k-module structure. One can define an action of t also, by declaring that

$$t(m \otimes n) = tm \otimes tn$$

for pure tensors $m \otimes n$. This gives $M \otimes_k N$ an $k[t]$-module structure. This construction gives a multiplication $*$ on $K_0(\operatorname{End}\mathcal{P}(k))$, which corresponds on generators to

$$(M, f) * (N, g) = (M \otimes_k N, f \otimes g).$$

The ring structure on $\tilde{K}_0(\operatorname{End}\mathcal{P}(A))$ was determined by Almkvist for $A = \mathbb{R}$ and $A = \mathbb{C}$. In this paper, we write down the ring structure for any field.

To make our exposition more self-contained, in §2, we review the calculation of the K-theory of finitely many commuting endomorphisms. Although this is well-known, the proof we give to our knowledge does not appear in the literature. In §3, we then describe the multiplication on $\tilde{K}_0(\operatorname{End}\mathcal{P}(k))$ and write down a few consequences.

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2. Review of the Category $\operatorname{End}_n \mathcal{P}(k)$

In this section, we provide one way to calculate the abelian groups $K_i(\operatorname{End}_n \mathcal{P}(k))$, a result which is well-known to experts.

Let $\operatorname{End}_n \mathcal{P}(k)$ denote the exact category of $k[T] := k[t_1, \ldots, t_n]$-modules that are finitely generated as k-modules. In this section we calculate $K_i(\operatorname{End}_n \mathcal{P}(k))$. A finite-dimensional k-vector space V with any n commuting endomorphisms $f_1, f_2, \ldots, f_n$ of V may also be considered as a $k[t_1, \ldots, t_n]/I$ module where I is the kernel of the map $k[t_1, \ldots, t_n] \to k[f_1, \ldots, f_n]$ given by $t_i \mapsto f_i$; hence:

2.1. **Proposition.** *The category* $\operatorname{End}_n \mathcal{P}(k)$ *is naturally equivalent to the filtered direct limit*

$$\varinjlim_I \,(k[T]/I\text{-}\mathbf{mod_{fg}}) \tag{1}$$

of exact categories, where I runs over the set of ideals of $k[T]$ such that the quotient $k[T]/I$ is finite-dimensional over k, and if $I \subseteq J$ are two such ideals, then the functor $k[T]/J$-$\mathbf{mod_{fg}} \to k[T]/I$-$\mathbf{mod_{fg}}$ is the forgetful functor induced by the quotient map $k[T]/I \to k[T]/J$.

Proof. The limit is indeed filtered, since $k[T]/(I \cap J)$ is finite-dimensional whenever $k[T]/I$ and $k[T]/J$ are finite-dimensional. Define a functor

$$F : \varinjlim_I \,(k[T]/I\text{-}\mathbf{mod_{fg}}) \to \operatorname{End}_n \mathcal{P}(A)$$

as follows. Any element in $\varinjlim_I (k[T]/I\text{-}\mathbf{mod_{fg}})$ is represented by $V \in k[T]/I\text{-}\mathbf{mod_{fg}}$ for some I. We let $F(V)$ to be the $k[T]$ module given by the forgetful functor induced by the map $k[T] \to k[T]/I$. This functor is well-defined because $k[T]/I$ is finite-dimensional, and it is easy to see that F gives the required natural equivalence. □

Using this observation and the result that taking K-groups of exact categories commutes with filtered direct limits [Qui72, §2], we obtain the following corollary.

2.2. **Corollary.** *The K-theory of the category $\mathrm{End}_n\,\mathcal{P}(A)$ may be calculated as the direct limit*

$$K_i(\mathrm{End}_n\,\mathcal{P}(A)) \cong \varinjlim_I K_i\left(k[T]/I\text{-}\mathbf{mod_{fg}}\right) \tag{2}$$

For any ring R, let $\mathrm{Jac}(R)$ denote the Jacobson radical of R. Any surjective ring homomorphism $R \to S$ induces a surjective ring homomorphism $R/\mathrm{Jac}(R) \to S/\mathrm{Jac}(S)$, and a commutative diagram of rings

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ R/\mathrm{Jac}(R) & \longrightarrow & S/\mathrm{Jac}(S) \end{array}$$

In turn, whenever S is finitely generated as an R-module, we have a commutative diagram of forgetful functors

$$\begin{array}{ccc} R\text{-}\mathbf{mod_{fg}} & \longleftarrow & S\text{-}\mathbf{mod_{fg}} \\ \uparrow & & \uparrow \\ R/\mathrm{Jac}(R)\text{-}\mathbf{mod_{fg}} & \longleftarrow & S/\mathrm{Jac}(S)\text{-}\mathbf{mod_{fg}} \end{array} \tag{3}$$

In particular, this applies to the ring homomorphisms $k[T]/I \to k[T]/J$ for $I \subseteq J$ appearing in the direct limit of (1).

2.3. **Proposition.** *The natural transformation of the direct limits of categories induced by the forgetful functors as in (3) for $R = k[T]/I$ induces an isomorphism algebraic K-groups*

$$\varinjlim_I K_i\left(\frac{k[T]/I}{\mathrm{Jac}(k[T]/I)}\text{-}\mathbf{mod_{fg}}\right) \xrightarrow{\sim} \varinjlim_I K_i\left(k[T]/I\text{-}\mathbf{mod_{fg}}\right).$$

Proof. For any Artinian ring R, the Jacobson radical $\mathrm{Jac}(R)$ is nilpotent (e.g. [Lam91, Theorem 4.12]), and so devissage [Qui72, §5, Theorem 4] shows that the inclusion $R/\mathrm{Jac}(R)\text{-}\mathbf{mod_{fg}} \to R\text{-}\mathbf{mod_{fg}}$ induces an isomorphism

$$K_i(R/\mathrm{Jac}(R)\text{-}\mathbf{mod_{fg}}) \to K_i(R\text{-}\mathbf{mod_{fg}})$$

(see [Wei13, Page 439] for more details). In particular, this applies to $R = k[T]/I$ where $k[T]/I$ is finite-dimensional over k. □

2.4. **Theorem.** *The algebraic K-groups of the exact category* $\mathrm{End}_n\,\mathcal{P}(k)$ *are given by*

$$K_i(\mathrm{End}_n\,\mathcal{P}(k)) \cong \bigoplus_M K_i(k[t_1,\dots,t_n]/M) \tag{4}$$

where M ranges over all the maximal ideals of the polynomial ring $k[t_1,\dots,t_n]$.

Proof. We must calculate the limit

$$\varinjlim_I K_i\left(\frac{k[T]/I}{\mathrm{Jac}(k[T]/I)}\text{-}\mathbf{mod}_{\mathbf{fg}}\right) \tag{5}$$

given in Proposition 2.3. So, fix an ideal I such that $k[T]/I$ is finite-dimensional. Then there are finitely many maximal ideals $M_1,\dots,M_k$ of $k[T]$ that contain I and

$$k[T]/I \cong \bigoplus_{j=1}^k (k[T]/I)_{M_j}$$

via the obvious map (e.g. [GW10, Theorem 5.20]). Here, by $(k[T]/I)_{M_i}$, we abuse notation and mean the localization of $k[T]/I$ away from the maximal ideal $M_i(k[T]/I)$. If J is an ideal containing I, then there is a subset $\{N_1,\dots,N_\ell\}$ of the maximal ideals $\{M_1,\dots,M_k\}$ that contain J and the quotient homomorphism $k[T]/I \to k[T]/J$ induces the map

$$\bigoplus_{j=1}^k (k[T]/I)_{M_j} \longrightarrow \bigoplus_{j=1}^\ell (k[T]/J)_{N_j}$$

which must be the projection map. The induced map on K-groups that fits into the direct limit in (5), by is then the map

$$\bigoplus_{j=1}^\ell K_i(k[T]/N_j) \longrightarrow \bigoplus_{j=1}^k K_i(k[T]/M_j).$$

given by the sum of the inclusion maps. Taking the direct limit gives the stated result, once we note that $k[T]/M$ is finite-dimensional over k for any maximal ideal M. □

In particular, we obtain the following amusing and well-known result.

2.5. **Corollary.** *For a field k, there is an isomorphism of abelian groups*

$$\tilde{k}_0 := \left\{\frac{1+a_1t+\cdots+a_nt^n}{1+b_1t+\cdots+b_mt^m} : a_i,b_j \in k\right\} \cong \bigoplus_{|\mathrm{Spec}(k[T])|-1} \mathbb{Z}.$$

3. The Ring Structure For One Endomorphism

In this section, we explore the ring structure on $\tilde{K}_0(\mathrm{End}\,\mathcal{P}(k))$ for various fields k other than the known results for k algebraically closed and $k=\mathbb{R}$. To start out investigation however, we give an alternative proof of the following theorem for algebraically closed fields:

3.1. **Proposition** ([Alm78, Proposition 3.4, Part 1])**.** *If k is an algebraically closed field then*

$$\tilde{K}_0(\operatorname{End}\mathcal{P}(k)) \cong \mathbb{Z}[k^\times]$$

where $\mathbb{Z}[k^\times]$ denotes the integral group ring for the multiplicative group $k^\times$ of k.

Proof. The group $\tilde{K}_0(\operatorname{End}\mathcal{P}(k))$ is the free abelian group $\bigoplus_{a\in k^\times} K_0(k[t]/(t-a))$. Suppose $a, b \in k^\times$. The generators for the direct summands corresponding to $K_0(k[t]/(t-a))$ and $K_0(k[t]/(t-b))$ are $[k]$ where t acts on k by a and by b respectively. The multiplication $[k] * [k]$ is then represented by $k \otimes_k k \cong k$ where t acts by ab, and so is the element corresponding to the generator $[k]$ for the summand $K_0(k[t]/(t-ab))$. Hence the map

$$\bigoplus_{a\in k^\times} \mathbb{Z} \longrightarrow \mathbb{Z}[k^\times]$$

$$(z_a)_{a\in k^\times} \longmapsto \sum_{a\in k^\times} z_a[a]$$

extends to an isomorphism $\tilde{K}_0(\operatorname{End}\mathcal{P}(k)) \to \mathbb{Z}[k^\times]$ as required. □

The above proof also shows that for any field k, the group ring $\mathbb{Z}[k^\times]$ is a subring of $\tilde{K}_0(\operatorname{End}\mathcal{P}(k))$, and the elements of this subring make up exactly the subgroup of $\tilde{K}_0(\operatorname{End}\mathcal{P}(k))$ consisting of those summands corresponding to the maximal ideals generated by polynomials of degree one. In particular, the ring structure on $\tilde{K}_0(\operatorname{End}\mathcal{P}(k))$ makes $\tilde{K}_0(\operatorname{End}\mathcal{P}(k))$ into a $\mathbb{Z}[k^\times]$-module.

3.2. **Proposition.** *The action of the ring $\mathbb{Z}[k^\times]$ on the abelian group $\tilde{K}_0(\operatorname{End}\mathcal{P}(k))$ making $\tilde{K}_0(\operatorname{End}\mathcal{P}(k))$ into a $\mathbb{Z}[k^\times]$-module can be described on generators by*

$$\left(a, \left[\frac{k[t]}{t^n - a_{n-1}t^{n-1} - \cdots - a_1 t - a_0}\right]\right) \mapsto \left[\frac{k[t]}{t^n - aa_{n-1}t^{n-1} - \cdots - a^{n-1}a_1 t - a^n a_0}\right]$$

for all $a \in k^\times$.

Proof. Indeed, the chosen representative on the right is isomorphic as a $k[t]$-module to

$$\frac{k[t]}{t-a} \otimes_k \frac{k[t]}{t^n - a_{n-1}t^{n-1} - \cdots - a_1 t - a_0}.$$

□

It is easy to see that $t^n - aa_{n-1}t^{n-1} - \cdots - a^{n-1}a_1 t - a^n a_0$ is irreducible whenever $t^n - a_{n-1}t^{n-1} - \cdots - a_1 t - a_0$ is irreducible, so that elements of the group $k^\times$ act as shift operators on $\tilde{K}_0(\operatorname{End}\mathcal{P}(k))$.

At this point, it seems more natural to work with the ring $\tilde{k}_0$ directly, though to be consistent with our description in terms of maximal ideals as in Theorem 2.4, we will use the $f \mapsto \det(\mathbf{1}t - f)$ version of the characteristic polynomial. So, we will abuse notation and write $\tilde{k}_0$ to mean

$$\tilde{k}_0 = \left\{ \frac{t^n - a_{n-1}t^{n-1} - \cdots - a_0}{t^m - b_{m-1}t^{m-1} - \cdots - b_0} : (a_i) \in (k^\times)^n, (b_i) \in (k^\times)^n \right\}.$$

Then, Proposition 3.2 tells us that the multiplication in $\tilde{k}_0$ of an element of the form $t-a$ with an arbitrary element can be described on generators by

$$(t-a) * (t^n - a_{n-1}t^{n-1} - \cdots - a_0) = t^n - aa_{n-1}t^{n-1} - \cdots - a^n a_0. \tag{6}$$

Naturally, with formula (6), one can also multiply higher degree generators together if one of them factors completely over k, since *usual polynomial multiplication* is addition in the ring $\tilde{k}_0$. One could define the product $f*g$ of two polynomials in $\tilde{k}_0$ by factoring one over a finite extension of k, and perform the multiplication there. Indeed, for any field k and an automorphism $\sigma : k \to k$ of fields, it is clear that $\sigma(f*g) = \sigma(f)*\sigma(g)$. Hence $f*g$ obtained in this way is still an element of $\tilde{k}_0$. This observation leads us to:

3.3. **Theorem.** *For any field k, there is an isomorphism $\tilde{K}_0(\operatorname{End}\mathcal{P}(k)) \cong \tilde{k}_0$, where $\tilde{k}_0$ is the ring whose underlying abelian group is free on the irreducible monic polynomials $f \in k[t]$ such that $f \neq t$, and whose multiplication is given for generators f and g by:*

(1) *If $f = t-a$ where $a \in k^\times$ and $g = t^n - a_{n-1}t^{n-1} - \cdots - a_0$ then*

$$f * g = t^n - aa_{n-1}t^{n-1} - \cdots - a^n a_0,$$

(2) *In general, if $f(t) = (t-e_1)(t-e_2)\cdots(t-e_k)$ where $(e_i) \in (E^\times)^n$ and E is a splitting field for f, then*

$$f * g = [(t-e_1)*g][(t-e_2)*g]\cdots[(t-e_k)*g].$$

Proof. The multiplication in (1) is Proposition 3.2. The multiplication rule in (2) follows because for each field extension E/k, the base change functor

$$\begin{aligned} \operatorname{End}\mathcal{P}(k) &\longrightarrow \operatorname{End}\mathcal{P}(E) \\ (V,f) &\longmapsto (V \otimes_k E, f \otimes 1) \end{aligned}$$

induces a homomorphism of rings when passing to $\tilde{K}_0$ that commutes with the isomorphism $\tilde{K}_0(\operatorname{End}\mathcal{P}(k)) \cong \tilde{k}_0$, and hence one can multiply after base change to a splitting field, using (1) for that splitting field. □

3.4. **Corollary.** *If k is perfect then there is an isomorphism of rings $\tilde{k}_0 \cong \mathbb{Z}[\bar{k}^\times]^{\operatorname{Gal}(\bar{k}/k)}$.*

Proof. An isomorphism is defined on generators by sending an irreducible monic polynomial f with factorisation $f(t) = (t-a_1)(t-a_2)\dots(t-a_n)$ over $\bar{k}$ to the element $\sum_{i=1}^n a_i \in \mathbb{Z}[\bar{k}^\times]^{\operatorname{Gal}(\bar{k}/k)}$. This is a ring homomorphism by Theorem 3.3, and it is bijective because over a perfect field, irreducible monic polynomials $f \in k[t]$ with $f(0) \neq 0$ correspond to the $\operatorname{Gal}(\bar{k}/k)$-orbits in $\bar{k}^\times$. □

Corollary 3.4 suggests the multivariable analogue. Recall that we use the notation $k[T] := k[t_1,\dots,t_n]$, and we let $\tilde{K}_0(\operatorname{End}_n\mathcal{P}(k))$ denote the nontrivial part of the group $K_0(\operatorname{End}_n\mathcal{P}(k))$. In other words, $\tilde{K}_0(\operatorname{End}_n\mathcal{P}(k))$ is the Grothendieck group of the category of n-commuting endomorphisms, none of which are zero. The following proposition is well-known.

3.5. **Proposition.** *Let k be a perfect field. If $M \subset k[T]$ is a maximal ideal, then $M\bar{k}[T] = M_1 \cap \cdots \cap M_d$ where $\{M_1, \ldots, M_d\}$ is the set of maximal ideals lying over M in $\bar{k}[T]$. The map*

$$M \mapsto \{M_1, \ldots, M_d\}$$

is a bijection between the maximal ideals of $k[T]$ and the Galois orbits in the set of maximal ideals of $\bar{k}[T]$.

Proof. This follows from [Mat86, Theorem 9.3,(iii)] and the fact that if $\{M_1, \ldots, M_d\}$ is a Galois orbit, then $(M_1 \cap \cdots \cap M_k) \cap k[T]$ is a maximal ideal of $k[T]$. □

3.6. **Theorem.** *If k is a perfect field then there is an isomorphism of rings $\tilde{K}_0(\mathrm{End}_n\,\mathcal{P}(k)) \cong \mathbb{Z}[(\bar{k}^\times)^n]^{\mathrm{Gal}(\bar{k}/k)}$.*

Proof. A generator of $\tilde{K}_0(\mathrm{End}_n\,\mathcal{P}(k))$ will be of the form $[k[T]/M]$ for some maximal ideal M. Proposition 3.5 says that $M\bar{k}[T] = \cap_{i=1}^d M_i$ where $\{M_1, \ldots, M_k\}$ are the maximal ideals lying over M. By Hilbert's Nullstellensatz each of these ieals are of the form

$$M_i = (t_1 - a_{i1}, \ldots, t_n - a_{in})$$

where $a_{ij} \in \bar{k}$. The map

$$\tilde{K}_0(\mathrm{End}_n\,\mathcal{P}(k)) \longrightarrow \mathbb{Z}[(\bar{k}^\times)^n]^{\mathrm{Gal}(\bar{k}/k)}$$

is then defined on generators by

$$[k[T]/M] \longmapsto \sum_{i=1}^{d} (a_{i1}, \ldots, a_{in}).$$

This is a ring homomorphism as in the Grothendieck group, direct sum and tensor product correspond to addition and multiplication, and Proposition 3.5 shows that it is a bijection. □

3.7. *Remark.* For imperfect fields, the maps to the Galois-invariant elements of group rings are still well-defined, but they are no longer surjective.

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